Quantum Morphisms

Lecture 8

Open problems

1) For what graphs G does it hold that
$G \xrightarrow{q} H \Rightarrow G \to H$ for all graphs H?
E.g. $G = C_n$. What about $G = K_4$? G planar?

2) For what graphs H does it hold that
$G \xrightarrow{q} H \Rightarrow G \to H$ for all graphs G?
E.g. H bipartite. Conjecture: No others.

3) Is $\chi_q(G) = 3$ & $\chi(G) > k$ possible
for any k?

4) Can we find examples of $G \xrightarrow{q} H$ but $G \not\to H$
where neither G nor H are complete?
Also don't want something of the form
$G \xrightarrow{q} K_n \xrightarrow{q} H$.

5) Is $\alpha_q(G) = \max\{\lfloor \gamma \rfloor : G$ has a proj. pack. of value $\gamma\}$?

6) $\alpha_p(G) \geq \sup\{\frac{1}{d}\alpha_q(dG) : d \in \mathbb{N}\}$

$$\geq \sup\{\tfrac{1}{d}\alpha_q(G[\overline{K_d}]) : d \in \mathbb{N}\}$$

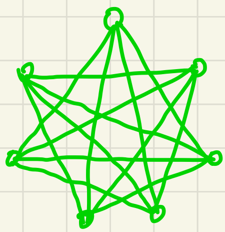
Are either of these equality?

7) Values of various quantum graph parameters on various (families of) graphs?

E.g. $\chi_q(K_{n:r}) = n-2r+2$? Kneser graphs

$\chi_f(K_{p/q}) = \frac{p}{q}$? Circular complete graphs

$K_{7/2}$:



Last Week

1) Defined (G,H)-isomorphism game.

- $G \cong H \iff \exists$ perfect classical strategy
- $G \cong_q H := \exists$ perfect quantum strategy
- $G \cong_{ns} H \iff \exists$ perfect non-signalling strategy

2) Defined quantum permutation matrix as

$\mathcal{P} = (P_{ij}) \in M_n(\mathbb{C}^{d\times d})$ s.t. $P_{ij} = P_{ij}^2 = P_{ij}^* \;\; \forall\, i,j \in [n]$

and $\sum_j P_{ij} = I = \sum_\ell P_{\ell k} \;\; \forall\, i,k \in [n]$.

3) Proved $G \cong_q H \iff \exists$ q-perm mtx $\mathcal{P} = (P_{gh}) \in M_n(\mathbb{C}^{d\times d})$

satisfying $\mathcal{P}^*(A_G \otimes I_d)\mathcal{P} = A_H \otimes I_d$

$\iff \quad (A_G \otimes I_d)\mathcal{P} = \mathcal{P}(A_H \otimes I_d)$

$\iff \quad P_{gh} P_{g'h'} = 0$ if $\mathrm{rel}(g,g') \neq \mathrm{rel}(h,h')$

4) $G \cong_q H \implies G$ & H cospectral

Very quick intro to C^*-algebras

Hilbert space – Real or complex inner product space that is a complete metric space with respect to the metric induced by the inner product.

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$;
- $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$;
- $\langle x, x\rangle \geq 0$ w/ equality only if $x = 0$.
- Induced norm: $\|x\| = \sqrt{\langle x, x\rangle}$
- Induced metric: $d(x,y) = \|x-y\|$
- Complete: every Cauchy sequence converges to a point in the space

Example 1: $\mathbb{C}^n$

Example 2 (Sequence spaces): S a countable set,

$$\ell^2(S) := \{(x_i)_{i\in S} : \sum_{i\in S} |x_i|^2 \text{ converges}\}$$

$$\langle x, y\rangle = \sum_{i\in S} x_i \overline{y_i}$$

Of particular interest: $S = \Gamma$ a group

Bounded Linear Operators

$\mathcal{H}$ - Hibert space

$B(\mathcal{H}) = \{ L : \mathcal{H} \to \mathcal{H} \mid L \text{ linear + bounded} \}$

- L bounded: $\exists\, \gamma \geq 0$ s.t $\forall x \in \mathcal{H}$

$$\|Lx\| \leq \gamma \|x\|.$$

$\iff L$ continuous.

- The infimum (minimum) such γ is the operator norm of L, $\|L\|_{op}$.

complex Hilbert space

C^*-algebra of operators: $\mathcal{A} \subseteq B(\mathcal{H})$ s.t.

- $X, Y \in \mathcal{A}$, $\lambda \in \mathbb{C} \Rightarrow \lambda X + Y \in \mathcal{A}$ + $XY \in \mathcal{A}$, i.e. $\mathcal{A}$ is an algebra over $\mathbb{C}$.
- $X \in \mathcal{A} \Rightarrow X^* \in \mathcal{A}$, where X^* is the adjoint of X

$$\langle X\psi, \varphi \rangle = \langle \psi, X^* \varphi \rangle \quad \forall \psi, \varphi \in \mathcal{H}$$

- $\mathcal{A}$ is closed in the operator norm:

$$X_n \in \mathcal{A},\ X \in B(\mathcal{H}) \;+\; \|X_n - X\|_{op} \to 0 \Rightarrow X \in \mathcal{A}$$

Example: $B(\mathcal{H})$ itself.

(Abstract) C^*-algebra:

An algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a norm $\|\cdot\|$ and map $^*: \mathcal{A} \to \mathcal{A}$ satisfying:

$$\left.\begin{array}{l}\bullet\ (x^*)^* = x \quad (^* \text{ is an involution}) \\ \bullet\ (x+y)^* = x^* + y^* \\ \bullet\ (\lambda x)^* = \bar{\lambda} x^* \\ \bullet\ (xy)^* = y^* x^*\end{array}\right\} \mathcal{A} \text{ is a } {*}\text{-algebra}$$

- $\|xy\| \leq \|x\| \|y\|$
- $\mathcal{A}$ is complete w/ respect to $\|\cdot\|$
- $\|x^* x\| = \|x\|^2$

<u>Example</u>: S a compact space, $C(S) := \{f : S \to \mathbb{C} \mid f \text{ continuous}\}$ is a commutative C^*-algebra under pointwise multiplication.

$\pi : \mathcal{A} \to \mathcal{A}'$ a linear map between $*$-algebras $\mathcal{A}$ & $\mathcal{A}'$.

π is a homomorphism if $\pi(xy) = \pi(x)\pi(y)$.

π is a $*$-homomorphism if <u>additionally</u> $\pi(x^*) = \pi(x)^*$.

GNS Theorem (Gelfand, Naimark, & Segal)

Let $\mathcal{A}$ be an abstract C^*-algebra. Then there exists a Hilbert space $\mathcal{H}$ & $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ s.t. $\|\pi(x)\| = \|x\| \;\; \forall\, x \in \mathcal{A}$.

Moreover, if $\mathcal{A}$ is unital (i.e. has a unit 1) then π can be chosen so that $\pi(1) = I_{\mathcal{H}}$.

The above conditions guarantee that π is injective. Thus π is a $*$-isomorphism between $\mathcal{A}$ and its image $\pi(\mathcal{A})$. Moreover the conditions on π ensure that $\pi(\mathcal{A})$ is a C^*-algebra of operators.

Thus every (abstract) C^*-algebra is $*$-isomorphic to a C^*-algebra of operators.

States on C^*-algebras

A state on a unital C^*-algebra $\mathcal{A}$ is a linear functional $s : \mathcal{A} \to \mathbb{C}$ such that $s(1) = 1$ and $s(x^* x) \geq 0 \quad \forall\, x \in \mathcal{A}$.

Example: if $|\psi\rangle \in \mathbb{C}^d$ is a unit vector, then

$s(M) = \langle \psi | M | \psi \rangle$ is a state on $\mathbb{C}^{d \times d}$.

$$\langle \psi | M^* M | \psi \rangle = \| M |\psi\rangle \|^2 \geq 0$$

A state s is faithful if $s(x^* x) = 0 \Rightarrow x = 0$.

A state s is tracial if $s(xy) = s(yx) \quad \forall\, x, y \in \mathcal{A}$.

Example: $tr(M) = \frac{1}{d} Tr(M) = \frac{1}{d} \sum_{i=1}^{d} M_{ii}$ is a faithful tracial state on $\mathbb{C}^{d \times d}$.

$$tr(M^* M) = \frac{1}{d} Tr(M^* M) \geq 0$$

GNS state Theorem: Let $s : \mathcal{A} \to \mathbb{C}$ be a state on a unital C^*-algebra. Then there is a Hilbert space $\mathcal{H}$, unit vector $|\psi\rangle \in \mathcal{H}$, and unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ s.t. $s(x) = \langle \psi | \pi(x) | \psi \rangle \quad \forall\, x \in \mathcal{A}$, and the subspace $\{ \pi(x) |\psi\rangle : x \in \mathcal{A} \}$ is dense in $\mathcal{H}$.

Quantum Commuting Strategies

Recall: A quantum tensor strategy for the (G,H)-isomorphism game consists of (let $V = V(G) \cup V(H)$)

1) unit vector $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ for some $d \in \mathbb{N}$;
2) POVMs $\mathcal{E}_x = \{E_{xy} \in \mathbb{C}^{d \times d} : y \in V\}$ $\forall x \in V$ for Alice;
3) POVMs $\mathcal{F}_x = \{F_{xy} \in \mathbb{C}^{d \times d} : y \in V\}$ $\forall x \in V$ for Bob.

This produces the correlation

$$p(y,y'|x,x') = \langle\psi| E_{xy} \otimes F_{x'y'} |\psi\rangle.$$

A quantum commuting strategy for the (G,H)-isomorphism game consists of (let $V = V(G) \cup V(H)$)

1) unit vector $|\psi\rangle \in \mathcal{H}$ for some Hilbert space $\mathcal{H}$;
2) POVMs $\mathcal{E}_x = \{E_{xy} \in B(\mathcal{H}) : y \in V\}$ $\forall x \in V$ for Alice;
3) POVMs $\mathcal{F}_x = \{F_{xy} \in B(\mathcal{H}) : y \in V\}$ $\forall x \in V$ for Bob.
4) satisfying $E_{xy} F_{x'y'} = F_{x'y'} E_{xy}$ $\forall x, y, x', y' \in V$.

This produces the correlation

$$p(y,y'|x,x') = \langle\psi| E_{xy} F_{x'y'} |\psi\rangle.$$

$G \cong_{qc} H$:= there is a perfect quantum commuting strategy for the (G,H)-isomorphism game.

From now on I will simply refer to this as quantum isomorphism.

$|\psi\rangle, E_{xy}, F_{x'y'}$ a perfect quantum tensor strategy

$\Rightarrow |\psi\rangle, E_{xy} \otimes I, I \otimes F_{x'y'}$ a perfect quantum commuting strategy.

strategy.

	finite	infinite
Tensor	q	qs
Commuting	q	qc

Theorem (Paulsen, Severini, Stahlke, Todorov, & Winter)

$G \cong_{qc} H$ if and only if there exists a unital C^*-algebra $\mathcal{A}$ with (faithful) tracial state $\tau : \mathcal{A} \to \mathbb{C}$, and projections $E_{gh} \in \mathcal{A}$ $\forall g \in V(G), h \in V(H)$ satisfying:

1) $\sum_h E_{gh} = 1 \quad \forall g \in V(G)$;
2) $\sum_g E_{gh} = 1 \quad \forall h \in V(H)$;
3) $E_{gh} E_{g'h'} = 0$ if $\mathrm{rel}(g,g') \neq \mathrm{rel}(h,h')$.

Theorem: $G \cong_{qc} H$ if and only if there exists a unital C^*-algebra $\mathcal{A}$ with a (faithful) tracial state, and a quantum permutation matrix $\mathcal{P} = (P_{gh}) \in M_n(\mathcal{A})$ such that

$$A_G \mathcal{P} = \mathcal{P} A_H$$

$M \in \mathbb{C}^{n \times n} \quad \mathcal{P} \in M_n(\mathcal{A})$

$(M\mathcal{P})_{ij} = \sum_k M_{ik} P_{kj}$

i.e. $\sum_{g' \sim g} P_{g'h} = \sum_{h' \sim h} P_{gh'} \quad \forall g \in V(G),\ h \in V(H)$

Later: remove tracial state requirement.